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ON A LINEAR TRANSFORMATION, AND SOME SYSTEMS OF HYPOCYCLOIDS[*]

BY R. E. ALLARDICE

IN the discussion of the envelope of the asymptotes of a system of similar conics through three fixed points,[†] I was led to the following equation in tangential coordinates:

$$\Sigma \sin^2 C(uv^2 \cos A + u^2 v \cos B) - 2(1 + \cos A \cos B \cos C)uvw$$
$$+ m(v \sin C - w \sin B)(w \sin A - u \sin C)(u \sin B - v \sin A) = 0, \qquad (1)$$

where m is a constant proportional to the cotangent of the angle between the asymptotes. It is easy to show that the curve represented by this equation has the straight line at infinity for a double tangent, the circular points being the points of contact. Hence the curve is of the fourth order and has three cusps; it is therefore a three-cusped hypocycloid, for this is the only three-cusped quartic that has the straight line at infinity for a double tangent, with the circular points as the points of contact.

The problem naturally suggested itself of transforming this equation to the form

$$(u + v + w)^3 - 27uvw = 0, \qquad (2)$$

where the triangle of reference is an equilateral triangle. This was accomplished by making use of the known forms of the tangential equations of the circular points at infinity, and using for the third equation that of the centre of the hypocycloid.

Thus we have the equations

$$\begin{aligned} ue^{ai} - ve^{(C+a)i} - we^{-(B-a)i} &= x + \omega y + \omega^2 z \equiv \lambda \text{ (say)}, \\ ue^{-ai} - ve^{-(C+a)i} - we^{(B-a)i} &= x + \omega^2 y + \omega z \equiv \mu, \\ p_1 u + p_2 v + p_3 w &= x + y + z \equiv \nu, \end{aligned}$$

where a, p_1, p_2, p_3 are constants to be determined, and ω is an imaginary cube root of unity. The equation (1) may be written in the form

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[*] Since some of the results in this paper are identical with some of the theorems given by H. A. Converse: "On a system of hypocycloids of class three inscribed to a given 3-line, and some curves connected with it" (ANNALS OF MATHEMATICS, second series, vol. 5, pp. 105–139, April, 1904), it is proper to state that the manuscript of the present paper was in the hands of the editors before Dr. Converse's article appeared.

[‡] ANNALS OF MATHEMATICS, ser. 2, vol. 3 (1901–1902), pp. 154–160.

$$l_1uv^2 + l_2uw^2 + m_1vw^2 + m_2vu^2 + n_1wu^2 + n_2wv^2 - 2k\,uvw = 0,$$

where the following relations subsist among the coefficients:

$$\begin{aligned} &l_1 \sin A \sin^2 B - l_2 \sin A \sin^2 C = m_1 \sin B \sin^2 C - m_2 \sin B \sin^2 A \\ &\quad = n_1 \sin C \sin^2 A - n_2 \sin C \sin^2 B = 2\,m \sin^2 A \sin^2 B \sin^2 C. \end{aligned} \tag{3}$$

Making use of the fact that the coefficients of u^3, v^3, and w^3 in (1) are zero, and employing the identical relation

$$(x + y + z)^3 - 27xyz = \lambda^3 + \mu^3 - 3\lambda\mu\nu,$$

it is easy to show that

$$p_1 = \tfrac{2}{3}\cos 3a, \quad p_2 = -\tfrac{2}{3}\cos(3C + 3a), \quad p_3 = -\tfrac{2}{3}\cos(3B - 3a);$$

and also that these values satisfy all the relations (3), except those involving m, whatever value a may have. From the last of these relations we find

$$m = \tan(C - B + 3a).$$

It is obvious from the form of equation (1) that the curve which it represents touches the sides of the triangle of reference; and, by varying the value of m, it is possible to make it represent any inscribed hypocycloid. Thus we may prove a number of theorems about a system of hypocycloids inscribed in a given triangle.

Locus of the Cusps. The tangential equations of the cusps are $x = 0$, $y = 0$, $z = 0$; hence the point-coordinates of one of the cusps, $x = 0$, are given by

$$\rho\xi = \frac{2}{3}\cos 3a + 2\cos a = \frac{8}{3}\cos^3 a,$$

$$\rho\eta = -\frac{2}{3}\cos(3C + 3a) - 2\cos(C + a) = -\frac{8}{3}\cos^3(C + a),$$

$$\rho\zeta = -\frac{2}{3}\cos(3B - 3a) - 2\cos(B - a) = -\frac{8}{3}\cos^3(B - a),$$

Eliminating ρ and a from these equations, we find for the locus of the cusps

$$a\sqrt[3]{\xi} + b\sqrt[3]{\eta} + c\sqrt[3]{\zeta} = 0,$$

or,

$$(a^3\xi + b^3\eta + c^3\zeta)^3 - 27a^3b^3c^3\xi\eta\zeta = 0.$$

This cubic curve has the sides of the triangle of reference as inflexional tangents, the straight line joining the points of inflexion being

$$a^3\xi + b^3\eta + c^3\zeta = 0.$$

Locus of the Centre. The tangential equation of the centre is $x + y + z = 0$; from which we easily obtain the point-equation of the locus of the centre in the form

$$\xi \sin 3A + \eta \sin 3B + \zeta \sin 3C = 0\cdot$$

This is a straight line passing through the nine-point centre $[\cos(B - C), \cos(C - A), \cos(A - B)]$ of the triangle of reference. The nine-point centre is the centre of the hypocycloid enveloped by the asymptotes of the equilateral hyperbola (a result due to Steiner).

I have also shown, in a paper on the envelope of the axes of a system of conics passing through three fixed points,* that the envelope of the axes of the system of conics referred to above is also a pair of hypocycloids. The tangential equation of one of them is

$$\begin{aligned}\Sigma\,[u(v\cos C + w\cos B - u)\,\{u\cos(B - C) - v\cos B - w\cos C\}]& \\ - \frac{2}{s+1}\,(v\cos C + w\cos B - u)(w\cos A + u\cos C - v)& \\ (u\cos B + v\cos A - w) &= 0.\end{aligned}$$

where s is the secant of the angle between the asymptotes.

The same transformation may be applied to this equation; and by making use of the relations

$$[u^3]\cos C + [u^2v] = [w^3]\cos A + [w^2v], \text{ etc.},$$

where $[u^3]$ denotes the coefficient of u^3 in the equation, I find for the values of the constants:

$$\begin{aligned} 3p_1 &= \rho\cos A - 2\cos(B - C), \\ 3p_2 &= \rho\cos B - 2\cos(C - A), \\ 3p_3 &= \rho\cos C - 2\cos(A - B), \\ \rho &= -4/s, \quad 3a = B - C. \end{aligned}$$

It may be shown that the hypocycloid represented by this equation touches the perpendicular bisectors of the sides of the triangle of reference;

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* *Transactions of the American Mathematical Society*, vol. 4 (1903), pp. 103–106.

and hence by varying s (or ρ) we have the equation of a system of hypocycloids touching three fixed *concurrent* straight lines.

The locus of the centre in this case is the straight line joining the circumcentre and the nine-point centre of the triangle of reference; and the centres of the two hypocycloids determined by one system of conics (of given eccentricity) are harmonically separated by these two points.

The locus of the cusps consists of three straight lines passing through the circumcentre; and the circle passing through the cusps touches two fixed straight lines passing through the circumcentre.

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